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On defining numbers of vertex colouring of regular graphs

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Abstract

In a given graph G, a set S of vertices with an assignment of colours to them is a *defining set of the vertex colouring of* G, if there exists a unique extension of the colours of S to a $\chi(G)$-colouring of the vertices of G.

A defining set with minimum cardinality is called a *minimum defining set* (of vertex colouring) and its cardinality, the *defining number*, is denoted by $d(G,\chi)$. Mahmoodian et al., have studied this concept. Here we study the defining numbers of regular graphs. Among other results the exact value of $d(n,r,\chi=r)$, the minimum defining number of all r-regular r-chromatic graphs with n vertices is determined, for $r=2,3,4$, and 5.

Keywords: Regular graphs; Colourings; Defining sets

1. Introduction

A *k-colouring* of a graph G is an assignment of k different colours to the vertices of G such that no two adjacent vertices receive the same colour. The (*vertex*) *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number k, for which there exists a k-colouring for G. A graph G with $\chi(G)=k$ is called a *k-chromatic* graph. In a given graph G, a set of vertices S with an assignment of colours to them is called a *defining set of vertex colouring*, if there exists a unique extension of the colours of S to a $\chi(G)$-colouring of the vertices of G. A defining set with minimum cardinality is called a *minimum defining set* (of a vertex colouring) and its cardinality is the *defining number*, denoted by $d(G,\chi)$. For example, in the case of a bipartite graph, this number is obviously equal to the number of connected components. For the Petersen graph P, $d(P,\chi)=4$. There are some results on defining numbers in [6].

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The concept of a defining set has been studied, to some extent, for block designs (see [9] for the references), and also under another name, a *critical set*, for latin squares (for example, see [1–3,7,8]). In [5] this concept is extended to graphs (see also [2,3]). Since graphical methods are used frequently in studying latin squares, this seems to be an appropriate concept. As the term '*critical*' is used in graph colourings in a different sense we choose to use the term '*defining*' which is also more suitable for this concept. Also note that as it is critical in this context to distinguish minimum from minimal we use the terminology 'minimum defining set' rather than 'minimal defining set' which is used in the case of block designs.

Defining sets of vertex colourings are closely related to the list colouring of a graph. In a list colouring for each vertex v there is a given list of colours $\mathscr{L}_v$ allowable on that vertex. Colouring must be done so that each vertex is coloured with an allowable colour and no two adjacent vertices receive the same colour. Indeed, it should be noted that any defining set S in a graph G naturally induces a list of possible colours for the vertices of the induced subgraph $\langle G-S\rangle$. Further, using this list of colours, $\langle G-S\rangle$ is uniquely list colourable. The following will be useful.

Definition (*Mahdian and Mahmoodian* [4]). A graph G with n vertices, is called a uniquely 2-list colourable graph, if for each vertex $v\in V$ there exists a list of colours on each vertex, $\mathscr{L}_v, |\mathscr{L}_v|=2$ such that there is a unique list colouring for G using this list.

Theorem A (Mahdian and Mahmoodian [4]). *A connected graph is uniquely 2-list colourable if and only if at least one of its blocks is not*

1. *a cycle,*
2. *a complete graph,*
3. *or a complete bipartite graph.*

If P is a graph theoretical property or equivalently a family of graphs, and d a numerical invariant of a graph, then the spectrum of P is

$$\mathrm{spec}(P,d)=\{n\mid \exists G, \text{ such that } P(G) \text{ is true and } d(G)=n\}.$$

First we study the question of possible spectrum of $d(G,\chi)$ for k-chromatic graphs on n vertices (Theorem 1). We then study the same question for regular graphs. To answer this, one first needs to know the possible values for the lower bound. Let $d(n,r,\chi=k)$ be the smallest value of $d(G,\chi)$ for all r-regular graphs with n vertices and the chromatic number equal to k. We discuss the value of $d(n,r,\chi=k)$ for certain parameters.

2. Spectrum of defining numbers

It is obvious that the defining number of a graph may depend on the number of vertices as well as on the chromatic number. So it is natural to try and find the spectrum of $d(G,\chi)$ for graphs G with $\chi(G)=k$ and $|V(G)|=n$. In the following theorem we answer this question.

Theorem 1. *Let P be family of graphs with $\chi(G)=k$ and $|V(G)|=n$. Then* $\mathrm{spec}(P, d(G,\chi))=[k-1, n-1]$.

Proof. In any k-colouring of G, each colour class contains a vertex for which there are $k-1$ different colours which appear in its neighbourhood. So if the colours of all vertices are given, except for one such vertex, then we can uniquely extend that colouring. Thus we have $d(G,\chi)\leqslant n-1$. Also the definition of the defining set implies $k-1\leqslant d(G,\chi)$. We have $d(K_k \cup \bar{K}_{n-k},\chi)=n-1$. For each $l\geqslant k$ we can construct a uniquely k-colourable graph G_l having l vertices and with $d(G_l,\chi)=k-1$. In fact, we can start with $G_k=K_k$ and suppose that $G_{l'}$ is constructed, then we add an extra vertex $v_{l'+1}$ by joining it to a subgraph K_{k-1} of $G_{l'}$. Now for each m, $k-1\leqslant m\leqslant n-1$, we have $d(G_{n-m-k-1}\cup \bar{K}_{m-k-1},\chi)=m$. The graph G_l is known as a k-tree. $\square$

If G is a k-chromatic graph, then by judiciously adding possible edges to G we can find a graph H with $d(H,\chi)=k-1$. In other words, any k-chromatic graph is a spanning subgraph of a graph H with $d(H,\chi)=k-1$. In fact, every k-chromatic graph G, is a spanning subgraph of a complete k-partite graph $K_{X_1,X_2,\ldots,X_k}$; where $X_1,X_2,\ldots,X_k$ are the colour classes of G and $d(K_{X_1,X_2,\ldots,X_k},\chi)=k-1$. The following theorem shows that any k-chromatic graph is also an induced subgraph of a graph H with minimum defining set of size $k-1$. First we recall the following definition.

Definition. Let G and H be two graphs, each with a given proper colouring with k colours. Then the chromatic join of G and H, denoted by $G\overset{\chi}{\vee}H$ is a graph where $V(G\overset{\chi}{\vee}H)$ is the union of $V(G)$ and $V(H)$, and $E(G\overset{\chi}{\vee}H)$, is the union of $E(G)$ and $E(H)$ together with the set $\{\{x,y\}\mid x\in V(G), y\in V(H)$ such that $\chi_G(x)\neq\chi_H(y)\}$.

The following theorem is straightforward.

Theorem 2. *Let G be a graph with $\chi(G)=k$ and $H=K_k\overset{\chi}{\vee}G$. Then $d(H,\chi)=k-1$ and G is an induced subgraph of H.*

3. Defining numbers of regular graphs

In this section we investigate the same question as in Section 2 for regular graphs. In the next theorem the spectrum of defining numbers is determined for r-regular 2-chromatic graphs.

Theorem 3. *Let P be the family of r-regular graphs with $\chi(G)=2$ and $|V(G)|=n$. Then*

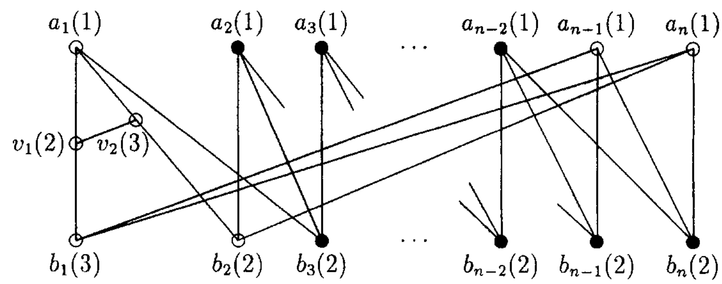
$$\mathrm{spec}(P, d(G,\chi=2)) = \left[1, \left\lfloor \frac{n}{2r} \right\rfloor\right].$$

Proof. We note that if p and n are both odd then $P=\emptyset$. If G is a bipartite graph with m components, then $d(G,\chi)=m$. Since every component of an r-regular bipartite graph has at least $2r$ vertices, $d(G,\chi=2) \leqslant \lfloor n/2r \rfloor$. Now for each m, $1 \leqslant m \leqslant \lfloor n/2r \rfloor$, we take the union of $m-1$ copies of $K_{r,r}$ with an r-regular bipartite graph of order $n-2(m-1)r$. The size of the smallest defining set of this graph is m. □

In contrast, we note that the size of defining sets may depend on the *specific* colouring of graphs. For example, the following theorem shows that the size of a defining set for a colouring could be very large (even if a defining set for a different colouring could be small).

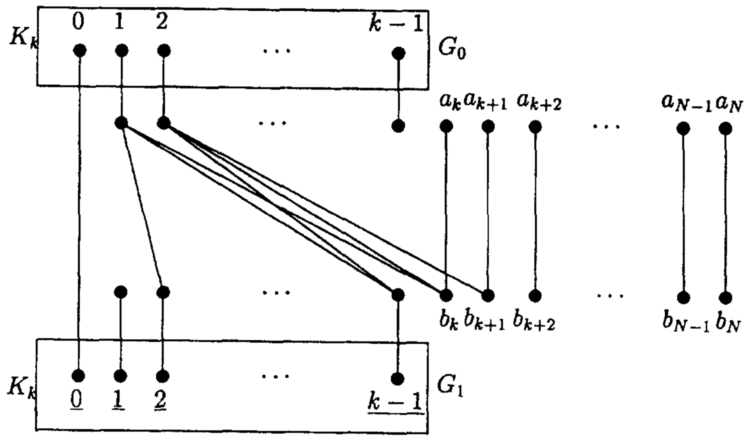
Theorem 4. *There exists a 3-chromatic 3-regular graph on $2n+2$ vertices with a colouring which needs at least $2n-4$ vertices in any defining set.*

Proof. We construct a graph G by the following process. First we construct a bipartite 3-regular graph H on the vertices $V(H)=\{a_1,a_2,\dots,a_n\}\cup\{b_1,b_2,\dots,b_n\}$, by joining a_i to b_j, where $j=i,i+1,i+2 \bmod n$. Then subdivide the edges a_1b_1 and a_1b_2 (respectively) by adding two vertices v_1 and v_2 on them (respectively), and joining v_1 and v_2. The resulting graph is 3-regular 3-chromatic, with $|V(G)|=2n+2$.



Now we define a 3-colouring c on G by $c(a_i)=1$, $i=1,2,\dots,n$, $c(b_j)=2$, $j=2,3,\dots,n$, $c(b_1)=c(v_2)=3$, and $c(v_1)=2$. In the figure the colour of each vertex is written in parenthesis. Any defining set of this colouring must contain the set $\{a_2,a_3,\dots,a_{n-3},b_3,b_4,\dots,b_n\}$ which has $2n-5$ elements. Since in the neighbourhood of each vertex of this set only one colour is present, it is easily seen that this set of vertices must be in any defining set and yet is not a defining set. But if we add, for example, $v_2(3)$ and $b_1(3)$ to this set then we obtain a defining set. □

There is an analogue for $k>3$. Let G_0 and G_1 be two copies of K_k on $0,1,2,\ldots,k-1$ and $\underline{0},\underline{1},\underline{2},\ldots,\underline{k-1}$, respectively, coloured by $c(i)=i$ and $c(\underline{i})=i+1 \bmod k$. Let H_N be the bipartite graph with $V(H_N)=A\cup B$, where $A=\{a_1,a_2,\ldots,a_N\}$ and $B=\{b_1,\ldots,b_N\}$ and $(a_i,b_j)\in E(H_N)$ for $i=1,2,\ldots,N$ and $j=i+1,i+2,\ldots,i+k-1 \bmod N$ and $(a_i,b_i)\in E(H_N)$ for $i>k-1$. Let $c(A)=0$ and $c(B)=1$. Let G (the crocodile eats the snake which is eating its own tail) be the graph which consists of G_0, G_1, H_N with the following additional edges: $\{0,\underline{0}\}$, $\{a_i,i\}$, $\{b_i,\underline{i}\}$, $i=1,2,\ldots,k-1$. The graph G is a k-regular, k-chromatic graph and a colouring on $2k+2N$ vertices which needs a defining set of at least $2N+2k-4$ vertices.



Our aim is to determine the minimum size of defining sets among all colourings. We concentrate on finding $d(n,r,\chi=k)$. Note that by a theorem of Brooks, for any r-regular k-chromatic graph, which has no component a complete graph or an odd cycle, we have $k\leqslant r$. In the following theorem a lower bound for the case of $r=k$ is obtained.

Theorem 5. *Let G be a k-regular k-chromatic graph with $|V(G)|=n$, and let S be a defining set for G. Then*

$$|S|\geqslant \left\lceil \frac{k-2}{2(k-1)}n+\frac{1+e}{k-1}\right\rceil,$$

where e is the number of edges of the subgraph induced by S.

Proof. First we show that the induced subgraph $\langle G-S\rangle$ is a forest. Suppose it contains a cycle. Let C be the smallest cycle in $\langle G-S\rangle$. We may assume that the colours of all vertices of G except the ones in C are given. Then the list of possible colours available for each vertex of C is at least of size 2. This contradicts Theorem A, as C, with this list of colours, is uniquely list colourable. Now as $\langle G-S\rangle$ is forest we have

$$|E(G-S)| \leqslant |V(G-S)| - 1,$$

or

$$\frac{kn}{2} - (k|S| - e) \leqslant n - |S| - 1.$$

It follows that

$$|S| \geqslant \left\lceil \frac{k-2}{2(k-1)}n + \frac{1+e}{k-1} \right\rceil. \quad \square$$

Corollary. *Theorem 5 implies that*

$$d(n,k,\chi=k) \geqslant \left\lceil \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)} \right\rceil.$$

Proof. Let G be a k-regular k-chromatic graph and S be a defining set for G. Then as in the proof of Theorem 5 the induced subgraph $\langle G-S\rangle$ is a forest and therefore it is 2-colourable. So the chromatic number of the subgraph induced by S must be at least $k-2$. Since it has at least one edge between every two colour classes, it has at least $\binom{k-2}{2}$ edges in total, i.e. $e \geqslant \binom{k-2}{2}$. $\square$

Now we investigate the cases when the equality holds in the above corollary. First we state two lemmas. By applying these lemmas we can recursively construct graphs for which the equality holds in the corollary. But first a definition.

Definition. A defining set S with an assignment of colours in graph G, is called *a strong defining set*, if there exists an ordering $\{v_1, v_2, \ldots, v_{n-s}\}$ of the vertices of $\langle G-S\rangle$ such that, in the induced list of colours in each of the subgraphs $\langle G-S\rangle$, $\langle G-S\cup\{v_1\}\rangle$, $\langle G-S\cup\{v_1,v_2\}\rangle, \ldots,$ and $\langle G-S\cup\{v_1,v_2,\ldots,v_{n-s}\}\rangle$, there exists at least one vertex whose list of colours is of cardinality 1.

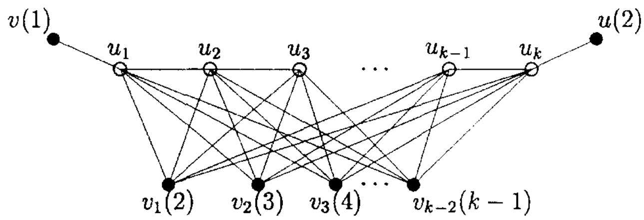
It should be noted that a similar concept is used to Keedwell [3] and is called 'strongly completable' and in the case of latin squares Cooper et al. [1] use the term 'strong critical set'.

Lemma 1. *Any defining set of a k-regular k-chromatic graph is strong.*

Proof. Let S be a defining set with an assignment of colours. Then as in the proof of Theorem 5, the induced graph $\langle G-S\rangle$ is a forest. Thus by Theorem A it is not a uniquely 2-list colourable graph. This implies that on the list of colours induced by S on $V(G-S)$, there must be a vertex with a list of size 1. Now the assertions follow by induction. $\square$

Lemma 2. *Let G be a k-chromatic k-regular graph with n vertices. Then there exists a k-regular graph H with $n+2(k-1)$ vertices, and a set $S' \subset V(H)$ of size $d(G,\chi)+(k-2)$ with assignment of colours, for which there exists a unique extension of colours of S' to a k-colouring of H.*

Proof. Let S be a defining set of size $d(G,\chi)$ for G. By Lemma 1, S is a strong defining set. Let v be the vertex in $\langle G-S \rangle$ which is the last vertex being coloured (see the definition of a strong defining set). There are $k-1$ different colours in the neighbourhood of v. And there are two vertices in its neighbourhood with the same colour. Let u be one of these vertices. We further assume that the colours of u and v are 2 and 1, respectively. Now we delete the edge uv, add $2(k-1)$ vertices $\{v_1,\ldots,v_{k-2},u_1,\ldots,u_k\}$ to G, and join them with the following procedure. Each of vertices $u_1,\ldots,u_k$ is joined to all of the vertices $v_1,\ldots,v_{k-2}$. Also u_i is joined to u_{i+1}, for $i=1,2,\ldots,k-1$. Moreover u_1 is joined to v and u_k to u. Now it can easily be checked that in the resulting graph, the set S with its colours in G together with vertices $\{v_1,\ldots,v_{k-2}\}$ and colouring $c(v_i)=i+1$, is a defining set. □



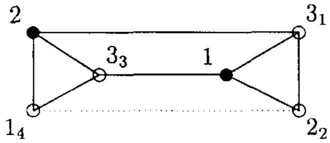
Theorem 6. *For $k=3,4$, and 5 we have*

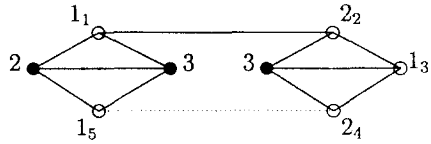
$$d(n,k,\chi=k)=\left\lceil \frac{k-2}{2(k-1)}n+\frac{2+(k-2)(k-3)}{2(k-1)} \right\rceil,$$

except possibly for $(n,k)=(10,5)$.

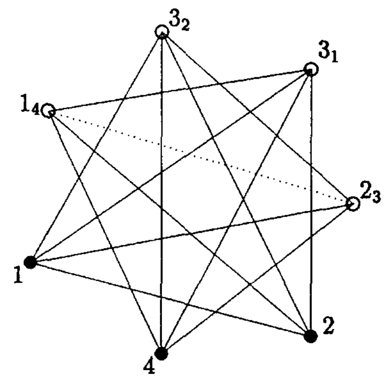
Proof. For each case, first we give some small graphs satisfying the formula, then we apply Lemmas 1 and 2 iteratively to obtain the theorem. Note that in each case the given graph is k-chromatic, even without the dotted edge, i.e. the edge to be deleted in the recursion. In the small cases the vertices of the defining set are shown by the filled circles and the symbols such as a_i indicate that the colour of that vertex is forced to be a at the ith step. In each step of recursion the edge being deleted is $u_k u$.

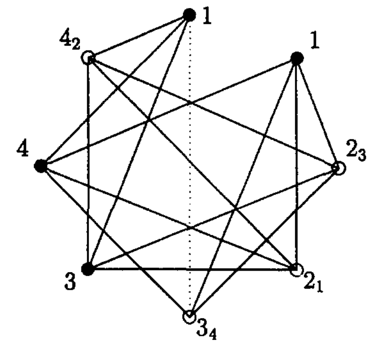
- $k=3$. For every 3-regular 3-chromatic graph the number of vertices n is even and $n \geqslant 6$. So the following graphs with 6 and 8 vertices serve our purpose.

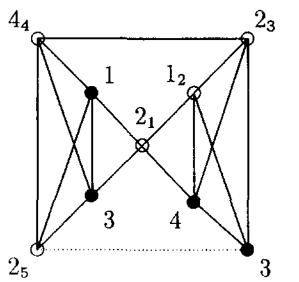


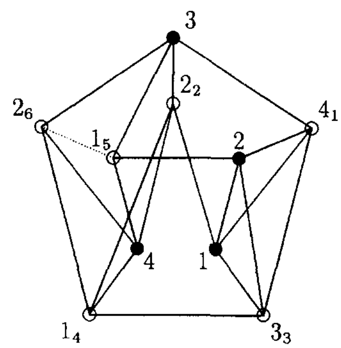


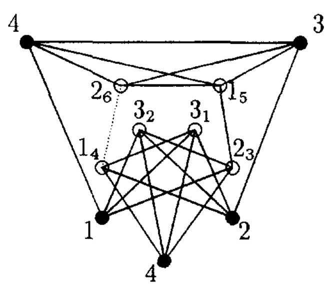
- $k=4$. In this case we must have $n \geqslant 7$ and the following 6 graphs with $7, \ldots, 12$ vertices are sufficient.

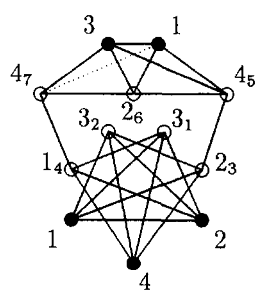




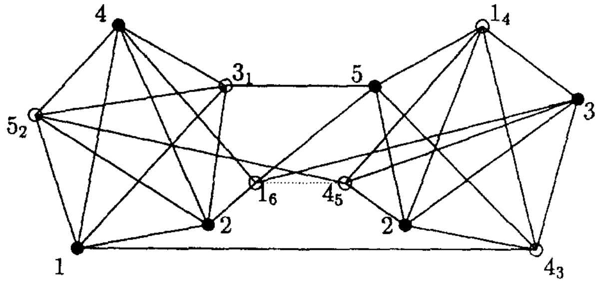








- $k=5$. It is easy to check that $n \geqslant 10$, and n must be even. We were not able to find a 5-regular 5-chromatic graph with 10 vertices with a defining set of size smaller than 6. But the following 4 graphs with 12, 14, 16 and 18 vertices are sufficient to find all others for $n \geqslant 20$.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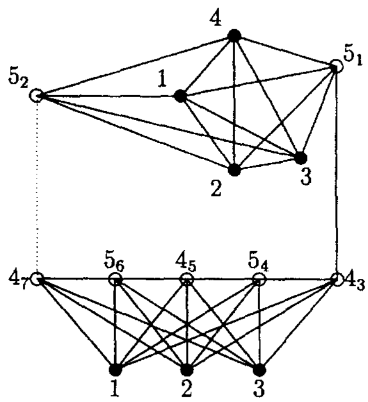
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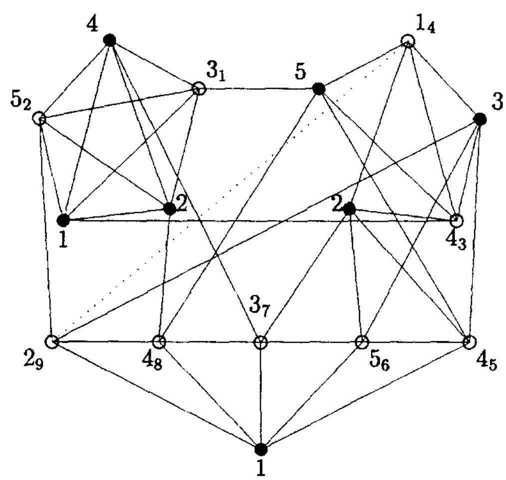
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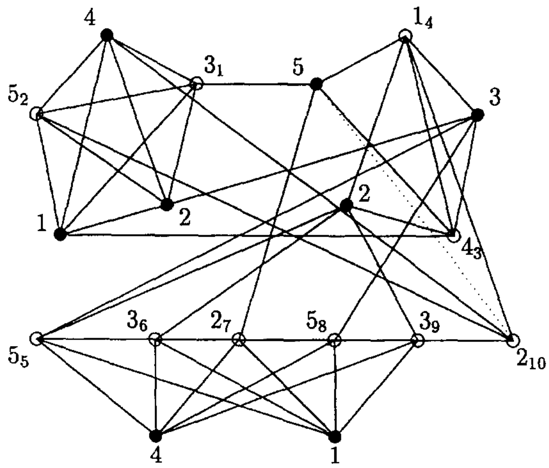

This completes the proof. □

When $r>k$, first we note the following.

Theorem 7. *Let n be a multiple of k, say $n=lk(l\geqslant 2)$; then*

$$d(n,2k-2,\ \chi=k)=k-1.$$

Proof. Obviously $d(n,2k,\ \chi=k)\geqslant k-1$. To show the equality, let $n=lk$. We construct a $(2k-2)$-regular k-chromatic graph with n vertices, as follows. Let $\{G_1,G_2,\dots,G_l\}$ be graphs such that $G_1=G_l=K_k$ and if $l\geqslant 3$ let $G_2=\cdots=G_{l-1}=\bar{K}_k$. Colour each G_i with k colours $1,2,\dots,k$. Then construct a graph G with lk vertices by taking the union of $G_1\cup G_2\cup\cdots\cup G_l$, and by making a chromatic join between G_i and G_{i+1}; for $i=1,2,\dots,l-1$. This is the desired graph. □

The following theorem with Theorem 6 determines $d(n,r,\ \chi=3)$, for all possible values of r.

Theorem 8. *For each n and each $r\geqslant 4$ we have,*

$$d(n,r,\ \chi=3)=2.$$

Proof. Obviously $d(n,r,\ \chi=3)\geqslant 2$. First we show the assertion for $r=4$. By Theorem 7 we have $d(3l,4,\ \chi=3)=2$.

For $n=3l+1, l\geqslant 2$, we construct a graph H with n vertices and $d(H,\ \chi=3)=2$ as follows. In the graph G which was constructed in Theorem 7, let $V(G_1)=\{v_1,v_2,v_3\}$, $V(G_l)=\{u_1,u_2,u_3\},c(v_1)=c(u_1)=1$, and $c(v_2)=c(u_2)=2$. We add a new vertex x and join it to all four vertices v_1,v_2,u_1, and u_2, and delete the edges u_1u_2 and v_1v_2. Then $S=\{v_1,v_2\}$ is a defining set.

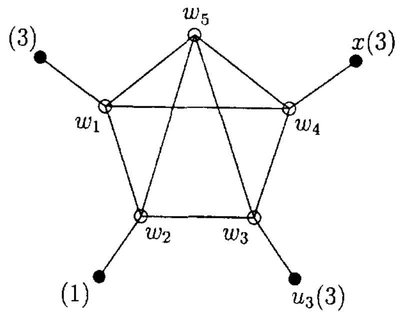

Fig. 1.

For $n=3l+2 (n \geqslant 11)$, we take the graph H, constructed for the case $n=3(l-1)+1$, and substitute the vertex u_2 by a graph F with 5 vertices $\{w_1, w_2, w_3, w_4, w_5\}$ such that $F=K_5 - \{w_1w_3, w_2w_4\}$. Then we join each of the 4 vertices w_1, w_2, w_3, w_4, to one of the different neighbourhoods of u_2 (see Fig. 1). For the resulting graph K, we have $d(K, \chi=3)=2$. (Actually the set $\{v_1, v_2\}$ is also a defining set here.)

For $n=8$, a graph $G=(V(G), E(G))$ where $V(G)=\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}$ and $E(G)=\{a_1b_2, a_1b_3, a_1c_1, a_1c_2, a_2b_1, a_2b_2, a_2b_3, a_2c_1, a_3b_1, a_3b_2, a_3b_3, a_3c_2, b_1c_1, b_1c_2, b_2c_1, b_3c_2\}$ can be taken which has the following colouring: $c(\{a_1, a_2, a_3\})=1$, $c(\{b_1, b_2, b_3\})=2$, and $c(\{c_1, c_2\})=3$. And $S=\{a_1, b_1\}$ is a defining set of size 2.

For $r=5$. We note that in $\bar{G}$, the complement of the constructed graph in Theorem 7 (for the case $n=3l$, l even), we have a 2-factor consisting of disjoint 6-cycles. So we see that $d(3l, r, \chi=3)=2$, for $5 \leqslant r \leqslant 6$. For the case $n=3l+1$ and $n=3l+2$, we use the same methods as in the case of $r=4$, adding extra suitable new vertices to so for our previous graphs. The case of $r \geqslant 6$ can be deduced from the cases $r=4$ and $r=5$ by applying a similar method. We leave the details to the reader. □

4. Conclusion

The following questions arise naturally from the work.

Question 1. *Is it true that for every k, there exists $n_0(k)$ such that for all $n \geqslant n_0(k)$ we have*

$$d(n, k, \chi=k) = \left\lceil \frac{k-2}{2(k-1)}n + \frac{2+(k-2)(k-3)}{2(k-1)} \right\rceil ?$$

Question 2. *Is it true that there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geqslant n_0(k)$ and $r \geqslant r_0(k)$ we have*

$$d(n, r, \chi=k) = k-1?$$

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